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Space-Time Singularities and the Kähler Cone

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Abstract: We review recent results on the interplay between the five-dimensional space-time and the internal manifold in Calabi-Yau compactifications of M-theory. Black string, black hole and domain wall solutions as well as Kasner type cosmologies cannot develop a naked singularity as long as the moduli take values inside the Kähler cone.

String and M-theory provide mechanisms and effects which improve on the problem of space-time singularities. Besides the resolution of singularities through α'-effects, the most interesting mechanism is what one might call 'the intervention of additional modes.' M-theory is a theory of extended objects, and winding states of strings and p-branes become massless for special values of the moduli parameterising the geometry of the compactified dimensions. When describing M-theory in terms of an effective action, it is crucial that *all* the light modes are taken into account. If one only includes those states which are massless away from the special points in the moduli space, then solutions with naked singularities are as generic as solutions without. However, in the context of the so-called enhançon geometry it was observed that naked singularities may be artifacts, which disappear when the dynamics of all relevant M-theory modes is taken into account [1]. It was soon realised that the same mechanism is at work in compactifications of M-theory on Calabi-Yau threefolds [2, 3]. In this note we review recent progress in establishing the absence of naked singularities for certain classes of five-dimensional space-times in a model-independent way, *i.e.*, valid for compactifications on arbitrary Calabi-Yau threefolds [4, 5]. The result of this work is that, due to a beautiful interplay between the internal manifold and space-time, electric and magnetic BPS solutions, BPS domain wall solutions, and Kasner cosmological solutions cannot develop naked singularities as long as the scalar fields take values in the interior of the M-theory moduli space, which in the case at hand is the so-called extended Kähler cone of the Calabi-Yau threefold. Moreover, the asymptotic

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behaviour of space-time geometries at the boundary of moduli space is determined by the type of degeneration of the internal manifold. Space-time singularities can occur when the boundary of the extended Kähler cone is reached, but in these cases the description in terms of a five-dimensional effective action breaks down, because infinitely many M-theory degrees of freedom become massless. Hence, these are not singularities of specific solutions, but rather indications that one needs a completely different description of the system.

The five-dimensional supergravity actions [6] considered here contain, besides the supergravity multiplet, n_V vector multiplets and n_H hypermultiplets. The action is fixed once the vector and hypermultiplet scalar manifolds have been specified and a gauging has been chosen. The vector multiplet manifold is a very special real manifold, *i.e.*, a hypersurface characterised by a homogeneous cubic polynomial, called the prepotential,

$$\mathcal{V}(X) := \tfrac{1}{6} C_{IJK} X^I X^J X^K \stackrel{!}{=} 1 \,, \tag{1}$$

where $I = 0, 1, \ldots, n_V$. The hypermultiplet manifold is quaternion-Kähler. We will consider the following cases: (i) actions without gaugings, where all multiplets are gauge-neutral and the scalar potential vanishes identically, and (ii) actions with a specific gauging, which correspond to Calabi-Yau compactifications with internal G-flux [7].

When a five-dimensional supergravity action is obtained by the compactification of eleven-dimensional supergravity on a Calabi-Yau threefold X, then it is fixed by the geometrical and topological data of the internal space [9]. We expand the Kähler form J of X in a basis of the second cohomology group

$$J = \mathcal{Y}^I \omega_I \,, \qquad \langle \omega_I \rangle = H^{1,1}(X) \,, \quad n_V = \dim H^{1,1}(X) - 1 \,, \tag{2}$$

with real moduli $\mathcal{Y}^I$. These moduli are related to the scalars X^I of eq. (1) by $\mathcal{Y}^I = V^{1/3} X^I$, where V is the volume of X. We choose a dual basis of four-forms, ν^I, two-cycles, C_I, and four-cycles, D_I, by Poincaré duality and intersection duality respectively, *i.e.,* $\int_X \nu^I \wedge \omega_J = \delta^I_J$, and $\int_{C^I} \omega_J = \delta^I_J$. The coefficients C_{IJK} of eq. (1) now have the interpretation of triple-intersection numbers, $C_{IJK} = D_I \circ D_J \circ D_K$, which implies that they are *integer valued* in contrast to generic five-dimensional supergravity actions where *real valued* constants are allowed.

The Kähler cone of X is defined by the requirement that all holomorphic curves $C \subset X$, surfaces $S \subset X$, and X itself have positive volume, *i.e.,* $\int_C J > 0$, $\int_S J \wedge J > 0$, and $V = \mathcal{V}(\mathcal{Y}) > 0$. We consider *primitive* boundaries, which are codimension-one boundaries where a single 2-cycle, $C^\star$, collapses. We call a basis *adapted*, if the Kähler cone takes the form $\mathcal{Y}^I = \int_{C^I} J = \mathrm{Vol}(C^I) > 0$. It is known that away from the so-called cubic cone the Kähler cone is locally polyhedral, whereas at the cubic cone the Calabi-Yau volume V vanishes. Therefore we can always choose such an adapted parameterisation, at least locally. In this basis primitive contractions correspond to blowing down one of the basic 2-cycles, $\mathcal{Y}^\star \to 0$, where $\star \in \{0, \ldots, n_V\}$.

At primitive boundaries of the Kähler cone, the following contractions can take place [10]:

- type I (“$2 \to 0$”): A finite number n of *isolated* curves in the homology class $C^\star$ is blown down to a set of points, $\mathrm{Vol}(C^\star) = \mathcal{Y}^\star \to 0$. This results in n charged hypermultiplets becoming massless.
- type II (“$4 \to 0$”): A divisor $D = v^I D_I$ collapses to a set of points: $\mathrm{Vol}(D) \propto (\mathcal{Y}^\star)^2$. This results in tensionless strings, implying that infinitely many particle-like states become massless.
- type III (“$4 \to 2$”): A (complex) one-dimensional family of curves sweeps out a divisor $D = v^I D_I$. Contracting this family of curves induces the collapse of D into a curve: $\mathrm{Vol}(D) \propto \mathcal{Y}^\star$. This results in $SU(2)$ gauge symmetry enhancement, possibly accompanied by a finite number of massless hypermultiplets in the adjoined representation.
- Cubic cone (“$6 \to 4$”, “$6 \to 2$”, “$6 \to 0$”): These contractions correspond to $V \propto \mathcal{Y}^\star$, $V \propto (\mathcal{Y}^\star)^2$ and $V \propto (\mathcal{Y}^\star)^3$. No M-theory interpretation is known, and it is clear that the description in terms of a five-dimensional effective action breaks down.

Boundaries of type I and type III can be crossed into the Kähler cone of a new Calabi-Yau threefold, which is birationally (and, for type III, even biholomorphically) equivalent to the original one. The extended Kähler cone is obtained by enlarging the Kähler moduli space at all boundaries of type I. Enlarging in addition the Kähler moduli space at all boundaries of type III results in the extended movable cone. However, this second extension only adds “gauge copies” to the parameter space (see [11, 3] for an explanation). While type I and type III boundaries are “internal boundaries” of the M-theory moduli space, type II contractions and the cubic cone lead to proper boundaries.

The metric on the Calabi-Yau Kähler moduli space is given by [12]

$$G_{IJ} := \frac{1}{2V} \int_X \omega_I \wedge \star\omega_J = -\frac{1}{2} \frac{\partial}{\partial \mathcal{Y}^I} \frac{\partial}{\partial \mathcal{Y}^J} \log \mathcal{V}(\mathcal{Y}) \; . \tag{3}$$

This metric is non-degenerate inside the Kähler cone. The link between curvature singularities of space-time and properties of the Kähler-cone metric is provided by the matrix

$$M_{IJ} = \frac{1}{2} \int_X J \wedge \omega_I \wedge \omega_J = \frac{1}{2}\, C_{IJK} \mathcal{Y}^K \; , \tag{4}$$

which is related to the Kähler-cone metric by [5]

$$\det(G_{IJ}) = -\frac{1}{2} \left(\frac{-1}{V} \right)^{\dim H^{1,1}(X)} \det(M_{IJ}) \; . \tag{5}$$

Since G_{IJ} is non-degenerate inside the Kähler cone, so is M_{IJ}.

It remains to analyse the behaviour of G_{IJ} (and M_{IJ}) at the primitive boundaries of the Kähler cone. As can be seen from eq. (5) $\det(G_{IJ})$ diverges at the cubic cone ($V = 0$). The three remaining types of boundaries (type I–III) are analysed as follows. Using eq. (5), for finite and non-zero Calabi-Yau volume V, we are able to infer regularity properties

type of boundary		physics	behaviour of $\det(G_{IJ})$
internal	type I	flop transition	regular
	type III	gauge symmetry enhancement	regular
external	type II	tensionless strings	zero
	cubic cone	unknown	divergent

Table 1: Behaviour of the Kähler moduli-space metric at the boundaries of the extended Kähler cone.

of the Kähler-cone metric G_{IJ} from the matrix M_{IJ} and vice versa: there is a one-to-one map of zero eigenvalues of G_{IJ} to zero eigenvalues of M_{IJ}. If $\det(M_{IJ})|_{\mathcal{Y}^\star\to 0} \propto (\mathcal{Y}^\star)^n$, then there are n linearly independent eigenvectors of M_{IJ} (and of G_{IJ}) satisfying

$$v^I_{(i)} M_{IJ}|_{\mathcal{Y}^\star\to 0} = 0 \ , \quad i = 1 \dots n \ . \tag{6}$$

Eq. (6) is supposed to hold throughout the face $\mathcal{Y}^\star = 0$. In particular, the null eigenvectors are determined by the triple intersection numbers, only. This implies that the components of the eigenvectors can be chosen to be *integer*. Hence, each zero eigenvector $v^I_{(i)}$ defines a divisor

$$D_{(i)} := v^I_{(i)} D_I \ . \tag{7}$$

If there is a holomorphic surface within the homology class $D_{(i)}$, then its volume vanishes always as $(\mathcal{Y}^\star)^2$ [5]. As a consequence, the divisors $D_{(i)}$, which are associated to null eigenvectors $v_{(i)}$, must perform a type-II contraction, where $\mathrm{Vol}(D) \propto (\mathcal{Y}^\star)^2$, rather than a type-III contraction, which is characterised by $\mathrm{Vol}(D) \propto \mathcal{Y}^*$. Thus we learn that the moduli-space metric is always regular at boundaries of type I and type III, while it develops a zero eigenvalue at boundaries of type II. Table 1 summarises our result, which is valid for *all* Calabi-Yau three-folds [5].

We now consider ungauged five-dimensional supergravity actions, which come from Calabi-Yau compactifications without flux or light brane winding states. These theories have string-like magnetic BPS solutions [13]

$$\mathrm{d}s^2 = e^{-U(r)}\Big\{ -\mathrm{d}t^2 + \mathrm{d}z^2 \Big\} + e^{2U(r)}\Big\{ \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{(2)} \Big\} \ . \tag{8}$$

Here, $e^{3U(r)} = \mathcal{V}(Y(r))$ is determined by the rescaled scalar fields $Y^I(r) := e^{U(r)} X^I(r) = e^{U(r)} V^{-1/3} \mathcal{Y}^I(r)$, which must be harmonic functions with respect to the transverse coordinates. For single-centered solutions this means $Y^I(r) = H^I(r) = c^I + \frac{p^I}{r}$, where r is the transverse radial coordinate, c^I are determined by the values of the moduli at transverse infinity, and p^I are the magnetic charges. The magnetic components of the gauge fields are proportional to $\partial_r Y^I(r)$. The Ricci scalar of this metric takes the form

$$R = -\, r^{-1} e^{-2U} \ \left(\tfrac{3}{2}\, r\, (U')^2 + 2\, r\, U'' + 4\, U' \right) \ . \tag{9}$$

Since other curvature invariants take a similar form, it is straightforward to show that for $r \neq 0$ singularities can only occur if either $e^U \to 0$ or if U' or U'' diverge. The latter can happen only if derivatives of the scalar fields Y^I diverge [3]. For $r \to 0$ one either

approaches a supersymmetric fixed point leading to a regular event horizon, so that the solution describes a supersymmetric black string [13], or the solution becomes singular at $r = 0$, which, after a suitable rescaling, can be treated as a limit of the discussion for $r \neq 0$. Using a parameterisation adapted to the Kähler cone, we immediately see that the solutions cannot become singular inside the Kähler cone, where $Y^I(r) > 0$ and all derivatives are bounded. The singularity $e^U \to 0$ corresponds to a particular limit of the cubic cone [3]: the overall volume V sits in a hypermultiplet, and therefore it is constant for magnetic BPS solutions. Eq. (1) shows that some of the $\mathcal{Y}^I$ go to zero, while others go to infinity, in such a way that the overall volume is kept constant.

Ungauged five-dimensional supergravity also has electric BPS solutions [14, 13]

$$\mathrm{d}s^2 = -e^{-4U(r)}\mathrm{d}t^2 + e^{2U(r)}\Big\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(3)}\Big\} \, . \tag{10}$$

Again, $e^{3U(r)} = \mathcal{V}(Y(r))$ can be expressed in terms of rescaled scalar fields $Y^I(r) = e^{U(r)}X^I(r)$ $= e^{U(r)}V^{-1/3}\mathcal{Y}^I(r)$, which this time have to satisfy the generalized stabilization equation

$$C_{IJK}Y^J(r)Y^K(r) = 2H_I(r) \, . \tag{11}$$

Here $H_I(r) = c_I + \frac{q_I}{r^2}$ are harmonic functions of the transverse coordinate, c_I are determined by the moduli at infinity, and q_I are the electric charges. The electric components of the gauge fields are proportional to $\partial_r(e^{3U(r)}Y^I(r))$ and the Ricci scalar takes the form

$$R = -r^{-1}e^{-2U}\Big(6\,r\,(U')^2 + 2\,r\,U'' + 6\,U'\Big) \, . \tag{12}$$

As the other curvature invariants again have the same structure, we see that curvature singularities can only occur if either $e^U \to 0$, which corresponds to a particular limit of the cubic cone, or if U' or U'' diverges. The latter only happens if derivatives of the scalar fields Y^I diverge. In general, one cannot solve eq. (11) for $Y^I(r)$ in terms of the harmonic functions. But fortunately, one does not need the explicit solution, because the occurrence of space-time singularities is controlled by the metric of the Kähler cone. To see this one uses that the quantities Y^I and $\mathcal{Y}^I$ are proportional. The factor of proportionality is an algebraic function of the H_I and Y^I, which is finite and non-vanishing for $r > 0$. By differentiating the generalized stabilisation equation (11) once, we obtain the relation $Y'^I = \frac{1}{2}\tilde{M}^{IJ}H'_J$, where $\tilde{M}_{IJ} = \frac{1}{2}C_{IJK}Y^K$ is a rescaled version of M_{IJ}. We can then use eq. (5) to make the connection to the metric of the Kähler cone. Similar arguments apply to the second derivatives of the moduli. As a consequence solutions are regular inside the Kähler cone and on type I and type III boundaries, whereas they become singular on type II boundaries and on the cubic cone. In a naive supergravity treatment one finds generic electric BPS solutions with naked singularities, where 'generic' means that one does not need to tune parameters to get a singular solution [2, 3]. The results of [5] guarantee that such singularities are artifacts, if they are not related to the cubic cone or to a type-II contraction. 'Faked singularities' result from not taking into account the threshold corrections of the states which become massless on internal boundaries (type I or type III) of the M-theory moduli space. This proves that the enhançon-like mechanism, which

was observed to be present in particular models in [2, 3] works for arbitrary Calabi-Yau threefolds.

For technical reasons, we assumed $r > 0$, so that the harmonic functions (and all their derivatives) are finite. If the limit $r \to 0$ is regular, the solution approaches a supersymmetric fixed point, which is the event horizon of an extremal black hole [14, 13]. If not, then the boundary of the extended Kähler cone is reached in this limit.

We now turn to BPS domain walls, which are solutions of gauged five-dimensional supergravity. We only consider the particular gauging which describes the bulk dynamics of five-dimensional heterotic M-theory, which can be obtained by dimensional reduction on a Calabi-Yau threefold with internal G-flux. The domain walls take the form [7]

$$\mathrm{d}s^2 = e^{2U(y)}\Big\{ -\mathrm{d}t^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\Big\} + e^{8U(y)}\mathrm{d}y^2 \ . \tag{13}$$

This time there are no gauge fields along the five-dimensional space-time, but the gauging induces a non-trivial scalar potential, which acts as a source of stress energy. As in the previous cases, $e^{3U(y)} = \mathcal{V}(Y(y))$ is determined in terms of rescaled scalar fields $Y^I(y) := e^{U(y)}X^I(y) = e^{U(y)}V^{-1/3}(y)\mathcal{Y}^I(y) = e^{-U(y)}\mathcal{Y}^I(y)$. Note that this time the hypermultiplet scalar V, which parameterises the overall volume of the Calabi-Yau space, is not constant, but given by $V(y) = \mathcal{V}(Y(y))^2 = e^{6U(y)}$. As in the electric case, the fields $Y^I(y)$ have to satisfy the generalized stabilization equation

$$C_{IJK}Y^J(y)Y^K(y) = 2H_I(y) \ , \tag{14}$$

where now $H_I(y) = b_I + a_I y$ are harmonic functions with respect to the single transverse coordinate. The Ricci scalar is

$$R = 4\, e^{-8U}\Big(3(U')^2 - 2U''\Big) \ . \tag{15}$$

Despite minor differences in details, the result of the analysis of curvature singularities is the same as for electric BPS solutions [5].

Let us finally discuss cosmological solutions of ungauged five-dimensional supergravity [4]. In contrast to the previous cases, these solutions are time-dependent and not BPS. For simplicity, we only consider five-dimensional flat FRW cosmologies, though the result also applies to more general Kasner solutions. It turns out to be convenient to keep a non-trivial lapse function:

$$\mathrm{d}s^2 = -e^{8U(t)}\mathrm{d}t^2 + e^{2U(t)}\Big\{(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 + (\mathrm{d}y)^2\Big\} \ . \tag{16}$$

Note that metrics of this type are related to domain walls (13) by a double Wick rotation. The Ricci scalar of this metric is given by

$$R = -4\, e^{-8U}\Big(3\dot{U}^2 - 2\ddot{U}\Big) \ . \tag{17}$$

With this particular choice of lapse function, the scalar equation of motion becomes the standard geodesic equation on the scalar manifold,

$$\ddot{\Phi}^\Sigma + \Gamma^\Sigma_{\Lambda\Theta}\dot{\Phi}^\Lambda\dot{\Phi}^\Theta = 0 \ , \tag{18}$$

with time t as affine parameter. Here Φ^Σ denotes all scalar fields, both from vector and from hypermultiplets. Since there is no scalar potential, the only source of stress-energy is the kinetic energy $T = \frac{1}{2} g_{\Sigma\Lambda} \dot{\Phi}^\Sigma \dot{\Phi}^\Lambda$, where $g_{\Sigma\Lambda}$ is the direct sum of the metrics of the vector and hypermultiplet scalar manifolds. In this case the link between the space-time geometry and the moduli is provided by Friedmann's equation, $4\dot{U}^2 = T$, and $\ddot{U} = 0$, where the last equation is a direct consequence of the conservation of T along geodesics. The latter also implies that finite-energy solutions cannot reach a boundary associated with the cubic cone in finite time, as the derivative of the scalar field associated with the diverging eigenvalue of the metric has to vanish asymptotically. Note that this statement does not only apply to the time coordinate t in (16), but also to the cosmological time τ, $\mathrm{d}\tau = e^{4U(t)}\mathrm{d}t$. Moreover, solutions are manifestly non-singular, as long as the metric on the M-theory moduli space is regular. The behaviour of solutions on all types of boundaries is discussed in detail in [4]. There we also consider cosmological solutions of a particular type of gauged five-dimensional supergravity [8], which is obtained by "integrating in" the charged hypermultiplets which become massless in a flop transition.

Our results obtained for black string, black hole and domain wall solutions as well as Kasner cosmologies support the conjecture that there is a general mechanism, which avoids space-time singularities in M-theory compactifications through the interplay with the internal dimensions. A natural next step is to consider type II compactifications on Calabi-Yau threefolds, which amounts to adding α'-corrections to the setup considered in this paper. More ambitiously, one can try to obtain analogous results without specifying the space-time geometry explicitly. This requires to link space-time and moduli space through estimates and inequalities, rather than equalities. One immediate question is whether there is a link between energy inequalities in space-time and properties of the moduli space metric.

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